9/7/2022 Saul Robinson

The Modelling of continuous events

{num}=appendix reference
[num]=source reference
//For appendix references go to end of document for explanations.

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1. Intro:

As evidenced by me being able to write this real life is sadly for all physicists is not a single discrete moment in time. So how do we predict what will happen? Since the start of time humanity has asked “If I smash rock against rock what will happen”, a tradition kept up to this day by the good people at CERN. And while resolving forces is nice doesn’t really say what will happen, just says how it is.

2. Differential equations:

//note this section will be more sparsely sourced as it is just deriving equations

Just finding an equation for what you want to model is one idea. Though it is easier said then done. But one important way such equations can be derived is by expressing what the rates of change would be for a single snapshot of time then working to a full equation. Though considering how much of university maths courses is solving partial differential equations it seems to be at times a… difficult endeavour.

Here are 3 examples in ascending difficulty.
Firstly a simple one

linear acceleration:

An object starts with velocity u and is accelerating linearly with a at time t how far how it moved.

$$welp\ a=\frac{d^2s}{dt^2}.\ So\int(a)\,dt=at+c=\frac{ds}{dt}$$

$$\frac{ds}{dt}=v\ so\ v=at+u$$

$$s=\int v\,dt=\frac{1}{2}at^2+ut+c\ suvat$$

Oh look one of our suvat equations.
What else of the A level physics syllabus be explained with these?

Harmonic motion:

I have a mass on the end of the spring which obeys Hooks Law. The mass is disturbed from it's rest position what will happen? First we need to build an equation to show what happens in a single discrete image of time, and then expand it out to an equation over time.

$$F=ma \;//\; F=ke$$

$kx=-ma$ ---> $a=\frac{d^2x}{dt^2}=-x\frac{k}{m}$ //negative ma as acceleration acts back towards rest position

(x here for x pos as we use e as in the constant later)

Good so we have our equation. $\frac{d^2x}{dt^2}=-x\frac{k}{m}$

Let $\frac{k}{m}=\omega^2$ As k and m are both positive we do. ω^2 And so. $\frac{d^2x}{dt^2}=-\omega^2 x$

$\frac{d^2x}{dt^2}+\omega^2 x=0$

With the general form for solving $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$ going to be shown below.

For the above to be valid for all values of 'x' the first and section derivative of the function must be eigenfunctions{1} so that a constant ratio between all the terms is kept.

One function that can fulfil this relation between the function and it's derivatives is: $y=Ae^{\lambda x}+Be^{\mu x}$

so $a(A\lambda^2 e^{\lambda x}+B\mu^2 e^{\mu x})+b(A\lambda e^{\lambda x}+B\mu e^{\mu x})+c(Ae^{\lambda x}+Be^{\mu x})=0$

$Ae^{\lambda x}(a\lambda^2+b\lambda+c)+Be^{\mu x}(a\mu^2+b\mu+c)=0$

For this to equal 0 for all values of t either the inside of each bracket equal 0 or $\mu=\lambda$ and $A=-B$.

But if that was to be true then $y=Ae^{\lambda x}+Be^{\mu x}=Ae^{\lambda x}-Ae^{\lambda x}=0$ // y=0 for all t.

So that possibility is not very useful. So instead let's have $\mu\neq\lambda\cup A\neq-B$

So the inside of each bracket is 0 so λ and μ must equal the roots of: $az^2+bz+c=0$

$z=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

Back to the case of our original equation $\frac{d^2x}{dt^2}+\omega^2 x=0$ we would have $z=\pm\frac{\sqrt{-4\omega^2}}{2}=\pm\sqrt{-\omega^2}=\pm i\omega$

And so $x=Ae^{i\omega t}+Be^{-i\omega t}$

using Eulers identity {2} $x=A(\cos(\omega t)+i\sin(\omega t))+B(\cos(\omega t)-i\sin(\omega t))$

As the x position of a spring obviously would not have an imaginary output the isins must cancel and so

$$B=A \text{ ---> } x=2A\cos(\omega t)$$

though if we add an $+\theta$ to the cos then it would not change this fulfilling the differential equation.

$$x=2A\cos\left(t\sqrt{\frac{k}{m}}+\theta\right)$$

Now as an extension what about those 2 constants the 2A and the θ. Welp a cosine curve goes between -1 and 1. And so the 2A must be the amplitude of the curve.

But to continue we need more variables:

$$x_0=\text{starting displacement} \;//\; v_0=\text{starting vel}$$

Welp the max possible displacement would be when all of the energy (kinetic+potential) is in the elastic potential energy store. $E_{k0}=\frac{1}{2}mv_0^2, E_{e0}=\frac{1}{2}kx_0^2$, ---> $x_{max}=\sqrt{\frac{0.5(mv_0^2+kx_0^2)}{0.5k}}=\sqrt{\frac{mv_0^2}{k}+x_0^2}$

At t=0 $x_0=x_{max}\cos(0+\theta), \theta=\arccos(\frac{x_0}{x_{max}})$

$$x=x_{max}\cos\left(t\sqrt{\frac{k}{m}}+\cos^{-1}(\frac{x_0}{x_{max}})\right),\; x_{max}=\sqrt{\frac{mv_0^2}{k}+x_0^2}$$

Heat:

Let's say that we had a thin metal rod so we only need to think in 1 dimension. And let's say that it is touching the outside world so the 2 end points of the rod are at fixed temperature which will be 0 degrees for simplicity's sake. Let's try and derive an equation that gives the temperature given an x position on the rod and a time. Well firstly let's describe how things act within a single moment.

To help think about this let's imagine our rod as a set of discrete points arranged in a line.

Each point is heated by those around it at a rate proportional to the difference in heat between them.

So each point in a single instant will be tending towards the mean temperature of the 2 points around it.

$$u = \text{temperature}, t = \text{time}, x = \text{x position}$$

$$u(x,t) = \text{our final function}, h = \text{distance, between points}, L = \text{length of rod}$$

The gradient between a point at it's neighbours would be about $\frac{d}{dx}(u(x \pm \frac{h}{2}, t))$. A positive gradient could either say that heat is flowing into or out of the point is looking at depending if it is to the left or right of point. And so we find the gradient to the right and then take the gradient from the left as to get the energy moving into our point because the magnitude of the gradient is proportional to the heat difference.

$$\frac{du}{dt} \propto \frac{\left(\frac{d}{dx}(u(x-\frac{h}{2},t)) - \frac{d}{dx}(u(x+\frac{h}{2},t))\right)}{h}$$

The $\div$h is a normalisation factor so that du/dt doesn't tend to 0 as h approaches 0.

$$\frac{du}{dt} \propto \lim_{h \to 0} \left[\frac{\left(\frac{d}{dx}(u(x-\frac{h}{2},t)) - \frac{d}{dx}(u(x+\frac{h}{2},t))\right)}{h} \right]$$

This is just the definition of differentiation{3} so can cay that this is the derivative of the function on top which is the derivative of u(x,t) and so we can it is the second derivative of u(x,t) with respect to x.

$$\frac{du}{dt} \propto \frac{d^2u}{dx^2} \text{ ---> } \frac{du}{dt} = k\frac{d^2u}{dx^2}$$

So now we can start thinking about our equation.

We need to find a function $u(x,t)$ which must:

Satisfy $\frac{du}{dt} = k\frac{d^2u}{dx^2}$

Be able to take any input for the initial temperature of the rod $u(x,0) = \phi(x)$

outside touching sides of rod $u(0,t) = 0 \cap u(L,t) = 0$

As the equation and the boundary conditions are linear and homogeneous. We can assume that:

$$u(x,t) = \sum_{1}^{\infty} (a_n * X_n(x) * T_n(t)) \{4\}$$

Where each individual term is a working solution to a specific cause which fulfils the boundary conditions.

$\frac{du}{dt} = \sum X(x) * \frac{du}{dt}$ $\frac{d^2u}{dx^2} = \sum T(t) * \frac{d^2X}{dx^2}$ I will be removing the sum from notation for conciseness.

Think back to our differential equation we derived earlier and differentiate it into the diff eq's form.

$$\frac{du}{dt} = k\frac{d^2u}{dx^2}$$

$$X(x)\frac{dT}{dt} = k * T(t) * \frac{d^2X}{dx^2}$$

$$\frac{1}{T(t)} * \frac{dT}{dt} = k * \frac{1}{X(x)} * \frac{d^2X}{dx^2}$$

The interesting thing that the left hand side is a function of t and the left hand side is a function of X ,yet they are always equal so both sides must equal a constant.

From here there a 3 cases we have to think about.
1:constant is positive // 2: constant is 0 // 3: constant is negative

Scenario 1 pos constant:

$$\frac{1}{T(t)}*\frac{dT}{dt}=\frac{1}{X(x)}*\frac{d^2X}{dx^2}=\lambda^2$$

$\frac{1}{T(t)}*\frac{dT}{dt}=\lambda^2$ ---> $\frac{dT}{dt}=\lambda^2 T(t)$ Simple just a exponential. $T(t)=Ae^{\lambda^2 t}$

$\frac{1}{X(x)}*\frac{d^2X}{dx^2}=\lambda^2$ ---> $\frac{d^2X}{dx^2}=\lambda^2 X(x)$ Cannot be sin or cos as those would have negative constant for second derivative.

$$X(x)=B\cosh(\lambda x)+C\sinh(\lambda x)\{5\} \text{ for definition of } \sinh,\cosh$$

$$u(x,t)=\sum_{n=1}^{\infty}\left(e^{\lambda_n^2 t}(k_{n1}\cosh(\lambda_n x)+k_{n2}\sinh(\lambda_n x))\right)$$

This is not a workable solution. With the positive exponential as $t\to\infty, u(x,t)\to\infty$ this would only not occur if $k_1=k_2=0$ so the rod have a initial temperature everywhere of 0 not a useful case.

Scenario 2 constant$=0$:

$$\frac{1}{T(t)}*\frac{dT}{dt}=\frac{1}{X(x)}*\frac{d^2X}{dx^2}=0$$

$\frac{dT}{dt}=0$ ---> $T(t)=A$

$\frac{d^2X}{dx^2}=0$ ---> $X(x)=Bx+C$

$$u(x,t)=\sum_{n=1}^{\infty}\left(k_{n1}x+k_{n2}\right)$$

For this to fulfil the boundary condition of $u(0,t)=0$ and $u(L,t)=0$ $k_{n1}=k_{n2}=0$.
So 0 heat along the whole rod again. Not very useful.

Scenario 3 negative constant:

$$\frac{1}{T(t)}*\frac{dT}{dt}=\frac{1}{X(x)}*\frac{d^2X}{dx^2}=-\lambda^2$$

$\frac{1}{T(t)}*\frac{dT}{dt}=-\lambda^2$ ---> $\frac{dT}{dt}=-\lambda^2 T(t)$ So negative exponential. $T(t)=Ae^{-\lambda^2 t}$

$\frac{1}{X(x)}*\frac{d^2X}{dx^2}=-\lambda^2$ ---> $\frac{d^2X}{dx^2}=-\lambda^2 X(x)$ So just a sin+cos curve. $X(x)=B\sin(\lambda x)+C\cos(\lambda x)$

$$u(x,t)=\sum_{n=1}^{\infty}\left(e^{-\lambda^2 t}(k_{n1}\sin(\lambda x)+k_{n2}\cos(\lambda x))\right)$$

For each solution to fulfil the =0 condition then all we need to do is $k_{n2}=0, \lambda=\frac{n\pi}{L}$

As $\sin(0)=0, \sin(n\pi)=0$

$$u(x,t)=\sum_{n=1}^{\infty}\left(a_n e^{-\lambda_n^2 t}\sin(\lambda_n x)\right)$$

So we have a workable solution but there is still a issue. We want to be able to put in any starting temperature distribution. $u(x,0)=\phi(x)=\sum_{n=1}^{\infty}\left(a_n\sin(\lambda_n x)\right)$

Now if you have heard of the Fourier series then I hope that light is buzzing in your head. Joseph Fourier
We need to be able to represent the input distribution as a sum of sine waves by finding the a_n constants.

For this we will use Sturm Lioville theorem {7} which goes as follows.

If $\frac{d^2y}{dx^2}+a(x)\frac{dy}{dx}+b(x)\,y=0$ can be written as:

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right]+\left[q(x)+\alpha r(x)\right]y=0 \text{ with}$$

$$k_1\frac{dy}{dx}+k_2\,y=0 \text{ at } x=a$$

$$k_3\frac{dy}{dx}+k_4\,y=0 \text{ at } x=b$$

Because with respect to differentiation y(x) is a eigenfunction{1} with the eigenvalue of alpha. Each function must have a different eigenvalue. $\alpha_n\neq\alpha_m$

be seen in the second y=0.

then: $\int_a^b (r(x)\,y_m\,y_n)\,dx=0$ // $y_m(x), y_n(x)$ are 2 different functions for y (must be eigenfunctions).

$\frac{d^2X}{dx}+\lambda^2 X=0$ This is a case o the Sturm Lioville start case $p(x)=1, q(x)=0, r(x)=1, \alpha=\lambda$

$$\frac{d^2X}{dx}+0*\frac{dX}{dx}+\lambda^2 X=0 \text{ ---> } \frac{d}{dx}\left[\frac{1*dX}{dx}\right]+\left[0+1\lambda^2\right]X=0$$

$$X_n(x)=\sin(\lambda_n x) \text{ where } \lambda_n=\frac{n\pi}{L}$$

$$X_m(x)=\sin(\lambda_m x) \text{ where } \lambda_m=\frac{m\pi}{L}$$

And our 2 points of $=0$ are at $x=0, x=L$ as that is part of the boundary conditions which we accounted for.

$$\int_0^L (r(x)X_nX_m)\,dx=\int_0^L (\sin(\lambda_n x)\sin(\lambda_m x))\,dx=0$$

Now remember back to the initial temperature distribution function.

$$\phi(x)=\sum_{n=1}^{\infty}(a_n\sin(\lambda_n x))$$

$$\int_0^L (\phi(x)\sin(\lambda_m))\,dx=\int_0^L\left[\sum_{n=1}^{\infty}(a_n\sin(\lambda_n x)\sin(\lambda_m x))\right]dx$$

A sum is just a concise way of saying add all of these. The integral of a bunch of terms summed is the same as integrating each term then adding. And so we can switch the integral and the sum around.

$$\int_0^L (\phi(x)\sin(\lambda_m))\,dx=\sum_{n=1}^{\infty}\left[\int_0^L (a_n\sin(\lambda_n x)\sin(\lambda_m x))\,dx\right]$$

As you know from our result from the Sturm Lioville theorem the integral on the right will be equal to 0 for $\lambda_n\neq\lambda_n$. Therefore: $\lambda_n=\lambda_m$

$$\int_0^L (\phi(x)\sin(\lambda_m))\,dx=\sum_{n=1}^{\infty}\left[\int_0^L (a_n\sin^2(\lambda_m x))\,dx\right]$$

$$\int_0^L (\phi(x)\sin(\lambda_m))\,dx=\int_0^L (a_n\sin^2(\lambda_m x))\,dx=a_n\frac{L}{2} \text{ // } \sin^2(x) \text{ integrates to } -0.25\cos(2x)+0.5$$

AANND SO:

$$u(x,t)=\sum_{n=1}^{\infty}\left(a_n e^{-\lambda_n^2 t}\sin(\lambda_n x)\right)$$

$$a_n=\frac{2}{L}\int_0^L (\phi(x)\sin(\lambda_n x))\,dx$$

3. that was hard let's just brute force it:

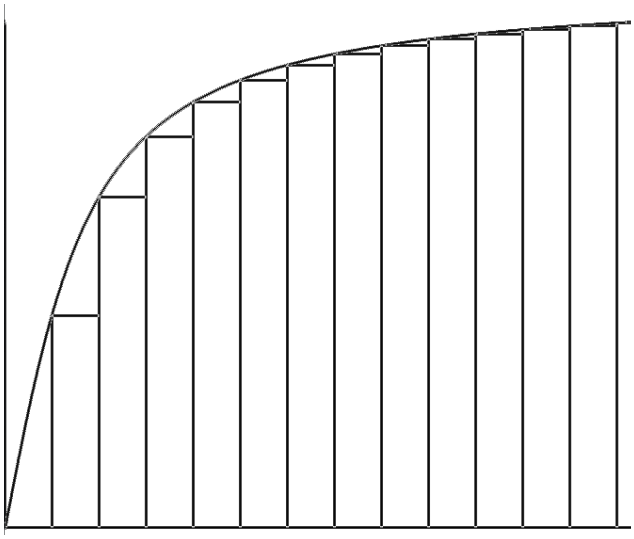
As seen there solving OED's and PDE's {6} can be difficult. And so a lot of these equations are left unsolved. In cases like these we could just take a bit more of a brute force approach. Take your continuous event and split it into a discrete set of timesteps. Upon each timestep you can do different things to get the value of the variables. If you have a simulation of an object such as the rod in the previous example you can represent it as a set of points in space.

The Euler Method:

This is essentially just what you do when you estimate the area under a curve with rectangles.

where $\frac{dy}{dt} = f(t, y(t))$ $h =$ timestep

$$y_{(n+1)} = y_{(n)} + hf(t, y_n)$$

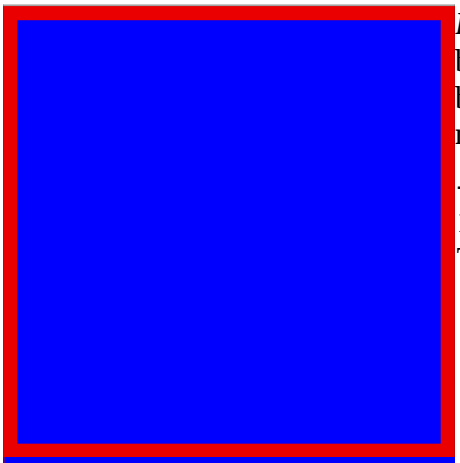


For the Euler method $\text{error} \propto \text{timestep\_size}$

This of course does slowly drift from the true value but it can still give simulations which can reflect real life.

It is also simple to imagine/implement. For example with this method for a let's say 2d simulation of heat transport all I would need is a grid of points with a starting temperature which each timestep would just compare their temperature to the temperature of the points around it and then transfer some of it's heat energy to those surrounding nodes proportional to the temperature difference.

//Here is a simulation I made which runs under such principles.



By the way:
blue=cold
black=middling
red=hot

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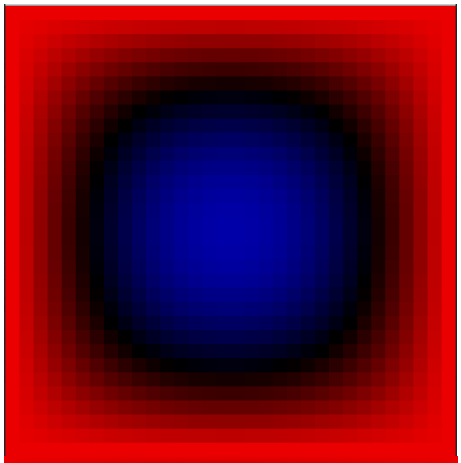
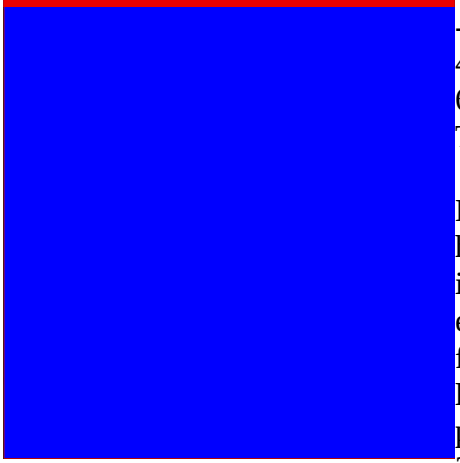
1000 timesteps 32 by 32 grid
Taking 0.01796. seconds





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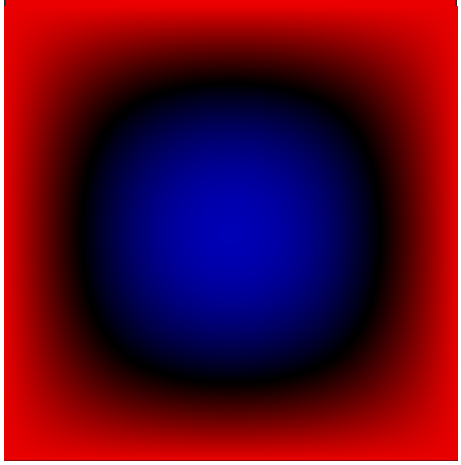
400,000 timesteps by
640 by 640
Taking 6415 seconds



Look at the heat equation and how the t has a lambda^2 next to it. Doubling the number of points essentially means half the frequency of sin waves and so lambda is halfed so for same progresstion you need 20^2*timesteps. $O(n^4)$.

The Runge Kutta method:

when $\frac{dy}{dt} = f(t, y), y(t_0) = y_0$

$$y_{(n+1)} = y_n + \frac{1}{6} h(k_1 + 2k_2 + 2k_3 + k_4)[2]$$

where

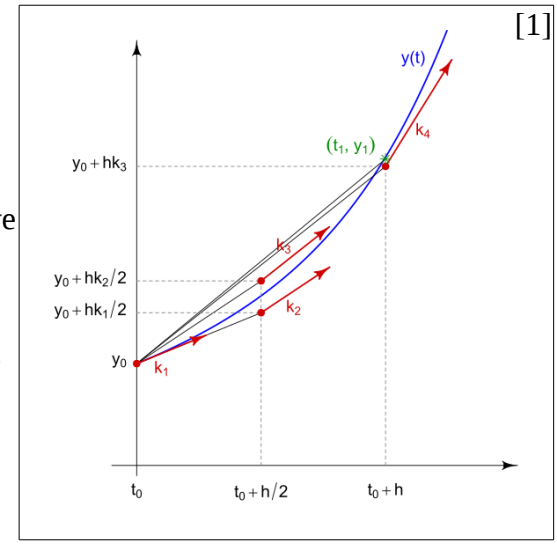
$$k_1 = f(t_n, y_n)$$

$$k_2 = f(t_n + \frac{h}{2}, y_n + k_1 \frac{h}{2})$$

$$k_3 = f(t_n + \frac{h}{2}, y_n + h\frac{k_2}{2})$$

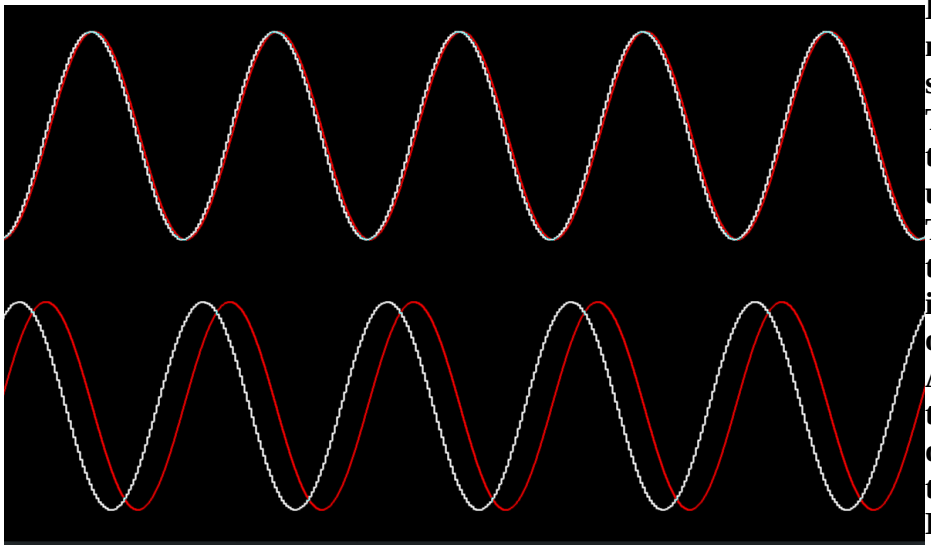
$$k_4 = f(t_n + h, y_n + hk_3)$$

Though this is really the name for a set of methods with different number of terms in the y(n+1) and with each k value being multiplied by different numbers.
Though sadly due to time constrains I will not be able to learn then explain the derivation of this 4 term equation which to my knowledge works by equation the equation with the k terms with unknown weights to the Taylor series expansion of the function.
If you want a more intuitive view of why this works here is a graph admittedly taken from wikipedia:



Though of course this method would be more computationally expensive then the Euler method it can give better approximations using the same timestep which in some cases could mean that with the same computational time you could get a better estimation.
[2] And a rough idea of the error can be found on the fly by comparing the outputs of runge kutta methods with a different number of steps. If there is too much error then whatever program your using can decrease the step size to try and return the error to a acceptable amount.

//On a more personal note to my reasoning I am not sure how valid mechanical methods of solving these equations can be for looking at how a system will act over a period of time compared to looking at what are the features of a system due to errors being able to slowly compound on themselves each iteration.



**Here is a simulation I made of a a mass spring system.
The red is the ground truth and the white is using the Euler method.
Top graph is the first 400 timesteps and the bottom is the last 400 timesteps of a total of 400,000.
As you can see by the end the Euler method has drifted out of sync with the true value.
H=0.25,k=0.1,m=1,s_0=1, v_0=0**

4. analogue computers:

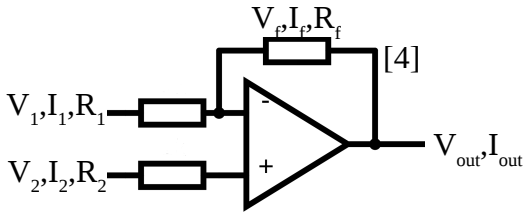
We have spent all this time trying to bend abstract and discrete systems to match continuous ones. But what if we just found something analogues to the original system. This technique has as of recent fallen out of favour due to the advent of digital computers which the only error comes from known variables such as how much precision the calculations are done to or how long the timestep is. This means that digital systems can do things such as having an adaptive timestep or just throwing more computing time at the problem if the solution is not accurate enough.

The Op-Amp:

Electrical analogue computers work around a component called the operational-amplifier or op-amp for short. The op amp has 2 inputs. 1 output. And an ideal op amp has infinite resistance.

$$V_{out}=-(\frac{R_f}{R_1}+1)(V_1-V_2)[3]\ ,$$

The R_I is for when $R_1=R_2=R_I$.



As can be seen from this first equation an op amp can be used to subtract 2 numbers, and if you do not want the effect of the *-1 then put R_f along the other terminal (the non-inverting terminal). While I do not understand the internals of a op-amp due to time one rule that explains op-amps well is that the op-amp will try to keep the 2 input pins at the same voltage by varying V_{out} [5]. This rule explains the following quite nicely.

Here I must touch on a odd thing I had to resolve mentally. There is 2 ways that V is thought about here. If V=IR then why does V_out exist without a resistor along with it?
Potential difference is the difference in energy per unit current. And so this can be measured by the difference over a component. But the electrons have a absolute amount of energy per unit charge (above ground). And this is what the op amp amplifies. With this V=IR does not really apply as V_out is referring to a fundamentally different quantity.

For simplicity's sake let $V_2=0$

$0=V_1-R_1I_1$op amp must now make other terminal 0 so over $R1$ all of V_1 must be lost.

$$I_1=\frac{V_1}{R_1}$$

Now to follow through to the V_out

$$V_{out}=V_1-I_1(R_1+R_f)=V_1-\frac{V_1}{R_1}(R_1+R_f)=V_1-V_1-\frac{R_f}{R_1}(V_1)=-\frac{R_f}{R_1}(V_1)$$

Now when $V_2\neq0$ then what we can do is to set V_2 as our new ground. As what absolute voltage is 0 is just an arbitrary point, then one mostly used ground is just what you get when you jam a metal rod into the ground. And so under this.

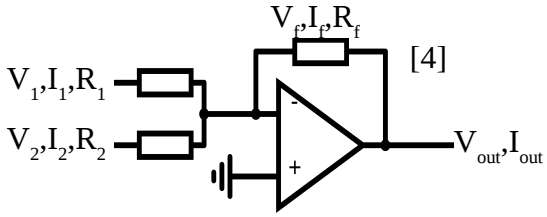
$$V_2=0,V_1=V_1-V_2(original)$$

And so:

$$V_{out}=-\frac{R_f}{R_1}(V_1-V_2)$$

//negative voltage is just the current moving in the opposite direction.

But what about adding:



[4]

For this we will use a similar line of logic to the regular op-amp configuration.
Each line over it's first resistor must loose enough current to go down to 0 as before so.

$$V_1 = I_1 R_{1,} V_2 = I_2 R_2$$

And as before no current flows through the op-amp.

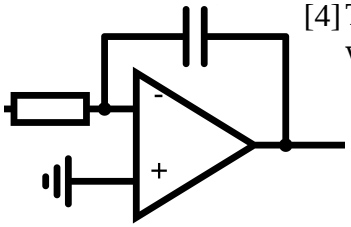
$$I_f = I_1 + I_2$$

Now we again say the drop over R_f is equal to V_{out}.

$$V_{out} = -R_f I_f = -R_f \left(\frac{V_1}{R_1} + \frac{V_2}{R_2}\right)$$

And if $R_2 = R_2$ then it simplifies to: $V_{out} = -\frac{R_f}{R_1}(V_1 + V_2)$.

Though one of the most useful things which you can do with a op-amp is do integration.[3]



[4] Though sadly I do not want to dedicate any more time to the project I will not be able to figure out why this works sadly.

Though sadly due to I guess it being easier and more versatile to throw more computing power at a problem then to improve a circuit analogue computing has fell out of fashion.
And that is all I really have to say erm, bye.

5. Appendix:

{1}: A function is a eigenfunction if when you apply it to another function the result is just something multiplied by the original function.

For example:

$$f(x)=e^{ax} \quad \frac{d}{dx}(e^{ax})=f'(x)=ae^{ax}=af(x)$$

$$af(x)=f'(x)$$

The function with respect to finding it's derivative is a eigenfunction and it's eigenvalue (the multiplier) is a.

$$e^{ix}=\cos(x)+i\sin(x) \text{ where } i=\sqrt{-1}$$

This can be derivative when you consider the Taylor series expansion of e^ix and that for e^x, cos and sin the Taylor series expansion is absolutely convergent

$$\frac{d}{dx}(e^{ix})=ie^{ix}$$

So the expansion is.

$$\frac{x^0}{0!}+i\frac{x^1}{1!}-\frac{x^2}{2!}-i\frac{x^3}{3!}+\frac{x^4}{4!}+i\frac{x^5}{5!}-\frac{x^6}{6!}-i\frac{x^7}{7!}+\frac{x^8}{8!}+\ldots$$

{2}: or splitting the even and odd terms

$$\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots\right)+\left(i\frac{x^1}{1!}-i\frac{x^3}{3!}+i\frac{x^5}{5!}-\ldots\right)$$

Which just so happens to be the Taylor series expansion of cos(x) +taylor series expansion of i*sin(x)

$$\cos(x)=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\ldots \;//\; \sin(x)=\frac{x^1}{1!}-\frac{x^3}{3!}+\frac{x^5}{5!}-\ldots$$

and so $e^{ix}=\cos(x)+i\sin(x)$

{3}: Definition of differentiation $f'(x)=\lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)$

{4}: Linear means that in the equation the function and it's derivatives just have constants in front of them (do not have functions applied to them). Homogeneous means that the right hand size is 0 essentially, the function and derivatives are just equal to 0.

Sadly I have not been able to find a good explanation as to why it works with most people going that the technique working in this example is proof enough of some kind of efficacy, I would **guess** that it needs to be linear homogeneous as each individual solution fulfils the differential equation and so summing all solutions would have a corresponding sum of of differential equations so if they all have a constant ratio between them then no matter the sum that ratio is maintained. And I would **guess** that they all need to =0 so then the sum of =0 no matter how large will still =0.

{5}: Hyperbolic cos and sin are odd functions seem to be related to what you would get from cos or sine if you put imaginary numbers in them.

$$\sinh(x)=\frac{e^x-e^{-x}}{2} \;//\; \cosh(x)=\frac{e^x+e^{-x}}{2}$$

$$\frac{d}{dx}\left(\frac{e^x-e^{-x}}{2}\right)=\frac{e^x-e^{-x}}{2} \;//\; \frac{d}{dx}\left(\frac{e^x+e^{-x}}{2}\right)=\frac{e^x+e^{-x}}{2}$$

{6}: ODE= ordinary differential equation, PDE: partial differential equation. The difference between them is that ODE's only contain derivatives with respect to 1 variables while PDE's contain derivatives with respect to multiple variables.

{7}:

for $\frac{d^2y}{dx^2}+a(x)\frac{dy}{dx}+B(x)\,y=0$

In an example which can be written as due to cancelling the dy/dx if with respect to differentiation y is a eigenfunction[1] with the eigenvalue of λ.

$$\frac{d}{dx}[p(x)\frac{dy}{dx}]+[q(x)+\lambda\, r(x)]\,y=0.$$

with at

$$x=a\,,k_1\frac{dy}{dx}+k_2\,y=0$$

$$x=b\,,k_3\frac{dy}{dx}+k_4\,y=0$$

./|\ ' $y_m\,y_n$ the 2 y functions must have different eigenvalues.

eq1: $\frac{d}{dx}[p(x)\frac{dy_m}{dx}]+[q(x)+\lambda_m r(x)]\,y_m=0$

eq2: $\frac{d}{dx}[p(x)\frac{dy_n}{dx}]+[q(x)+\lambda_n r(x)]\,y_n=0$

eq3 (eq1*yn): $y_n\frac{d}{dx}[p(x)\frac{dy_m}{dx}]+[q(x)+\lambda_m r(x)]\,y_m y_n=0$

eq4 (eq2*-ym): $-y_m\frac{d}{dx}[p(x)\frac{dy_n}{dx}]-[q(x)+\lambda_n r(x)]\,y_n y_m=0$

eq3+eq4: $y_n\frac{d}{dx}[p(x)\frac{dy_m}{dx}]+[q(x)+\lambda_m r(x)]\,y_m y_n-y_m\frac{d}{dx}[p(x)\frac{dy_n}{dx}]-[q(x)+\lambda_n r(x)]\,y_n y_m=0$

As you can see the bracket sets with q(x) cancel.

$$y_n\frac{d}{dx}[p(x)\frac{dy_m}{dx}]-y_m\frac{d}{dx}[p(x)\frac{dy_n}{dx}]+\lambda_m r(x)\,y_m y_n-\lambda_n r(x)\,y_n y_m=0$$

$$(\lambda_m-\lambda_n)(y_m y_n r(x))=y_m\frac{d}{dx}[p(x)\frac{dy_n}{dx}]-y_n\frac{d}{dx}[p(x)\frac{dy_m}{dx}]$$

$$(\lambda_m-\lambda_n)\int_a^b(r(x)\,y_n y_m)\,dx=\int_a^b\left(y_m\frac{d}{dx}[p(x)\frac{dy_n}{dx}]-y_n\frac{d}{dx}[p(x)\frac{dy_m}{dx}]\right)dx$$

$$.=\int_a^b(y_m\frac{d}{dx}[p(x)\frac{dy_n}{dx}])\,dx-\int_a^b(y_n\frac{d}{dx}[p(x)\frac{dy_m}{dx}])\,dx$$

.Let $I_1=\int_a^b(y_m\frac{d}{dx}[p(x)\frac{dy_n}{dx}])\,dx$ // Let $I_2=\int_a^b(y_n\frac{d}{dx}[p(x)\frac{dy_m}{dx}])\,dx$

To integrate I_1 and I_2 we use integration by parts. $\int(u(x)v'(x))\,dx=u(x)v(x)-\int(u'(x)v(x))\,dx$

$$I_1=y_m[p(x)\frac{dy_n}{dx}]_a^b-\int_a^b(\frac{dy_m}{dx}*p(x)\frac{dy_n}{dx})\,dx$$

$$I_2=y_n[p(x)\frac{dy_m}{dx}]_a^b-\int_a^b(\frac{dy_n}{dx}*p(x)\frac{dy_m}{dx})\,dx$$

$$(\lambda_m-\lambda_n)\int_a^b(y_m y_n r(x))\,dx=y_m[p(x)\frac{dy_n}{dx}]_a^b-\int_a^b(\frac{dy_m}{dx}*p(x)\frac{dy_n}{dx})\,dx-y_n[p(x)\frac{dy_m}{dx}]_a^b+\int_a^b(\frac{dy_n}{dx}*p(x)\frac{dy_m}{dx})\,dx$$

The 2 integrals cancel so.

$$(\lambda_m-\lambda_n)\int_a^b(y_m y_n r(x))\,dx=y_m[p(x)\frac{dy_n}{dx}]_a^b-y_n[p(x)\frac{dy_m}{dx}]_a^b$$ // more on next page

$$(\lambda_m-\lambda_n)\int_a^b (y_m y_n r(x))dx = y_m(b)p(b)y_n'(b)+y_n(b)p(b)y_m'(b)-y_m(a)p(a)y_n'(a)-y_n(a)p(a)y_m'(a)$$
$$.= p(b)[y_m(b)y_n'(b)-y_n(b)y_m'(b)]-p(a)[y_m(a)y_n'(a)-y_n(a)y_m'(a)]$$

Now firstly if $p(a)=p(b)=0$ then of course out right hand side would be zero but if this isn't the case then we need to think back to our original conditions.

$$x=a, k_1\frac{dy}{dx}+k_2 y=0 \quad // \quad x=b, k_3\frac{dy}{dx}+k_4 y=0$$
$$k_1 y_m'(a)+k_2 y_m(a)=0 \quad // \quad k_1 y_n'(a)+k_2 y_n(a)=0$$
$$k_3 y_m'(b)+k_4 y_m(b)=0 \quad // \quad k_3 y_n'(b)+k_4 y_n(b)=0$$

$$y_m'(a)=-\frac{k_2}{k_1}y_m(a) \quad // \quad y_n'(a)=-\frac{k_2}{k_1}y_n(a) \quad y_m'(b)=-\frac{k_4}{k_3}y_m(b) \quad // \quad y_n'(b)=-\frac{k_4}{k_3}y_n(b)$$

Substituting all o that into our erlier equation we get.

$$.= p(b)[-\frac{k_4}{k_3}y_m(b)y_n(b)+\frac{k_4}{k_3}y_n(b)y_m(b)]-p(a)[-\frac{k_2}{k_1}y_m(a)y_n(a)+\frac{k_2}{k_1}y_n(a)y_m(a)]$$

Inside each bracket all terms cancel and so.

$$(\lambda_m-\lambda_n)\int_a^b (y_m y_n r(x))dx=0$$

6. Sources:

[1]:
https://en.wikipedia.org/wiki/File:Runge-Kutta_slopes.svg
[2]:
https://www.google.com/url?sa=t&rct=j&q=&esrc=s&source=web&cd=&cad=rja&uact=8&ved=2ahUKEwjp5a-835v5AhUDolwKHbFwA70QFnoECCwQAQ&url=https%3A%2F%2Fmath.okstate.edu%2Fpeople%2Fbinegar%2F4513-F98%2F4513-l21.pdf&usg=AOvVaw3aaWE7VejwXONnckXtGVTq
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